Stochastic Privacy-Preserving Methods for Nonconvex Sparse Learning

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Abstract

In the big data era, sparse learning has become a very prevalent tool for mining useful information and patterns from high dimensional data in many applications. However, it has been shown that sensitive data may be inferred from learned sparse models by potential adversaries, leading to risks for security and privacy leakage. Although some pioneering works attempted to relieve such risks, they still confront the issue of heavy computational cost for large-scale problems. To address this issue, we propose two stochastic iterative hard thresholding (HT) methods that satisfy differential privacy (DP), which we name, DP-SGD-HT and DP-SCSG-HT. Employing the generalized Rényi differential privacy (RDP), formal and refined privacy analysis is provided for these algorithms. In DP-SGD-HT, stochastic gradient perturbation is adopted to release the shackles of heavy computational cost rooted in the calculation of full gradients in prior algorithms, which substantially reduces the computational complexity from $O(n\log(n))$ to $O(b\log(n))$, where b is the size of the mini-batch sample used to compute stochastic gradients, and n is the sample size. To further reduce the variance of perturbed stochastic gradient, DP-SCSG-HT performs stochastically controlled stochastic gradient perturbation by leveraging controlled variance reduction techniques. The computational complexity of DP-SCSG-HT is $O(\min\{1,\psi\}\cdot n\log n)$, where ψ can be much smaller than one in practice. We provide utility analysis of the proposed algorithms, which matches the best-known utility guarantee for nonconvex sparse optimization, while maintaining low computational complexity. Extensive experiments over real world medical and financial datasets demonstrate that our approaches outperform the DP baseline algorithm in terms of computational complexity.

Keywords: Sparse learning, differential privacy, stochastic algorithm

1. Introduction

Sparse learning that deals with high-dimensional data plays important roles in various data mining fields, such as bioinformatics [37] , image analysis [45], and engineering [46]. Many successful sparse learning applications for high dimensional problems rely on cardinality constraint for sparsity, which imposes challenges from both the statistical and computational analysis. In this paper, we consider the following nonconvex cardinality-constrained empirical risk minimization (ERM) problem,

$$\min_{x\in\mathbb{R}^d} f(x) := \frac{1}{n}\sum_{z=1}^{n} f_z(x) \quad \text{subject to} \quad \|x\|_0 \le k, \tag{1}$$

where $f(x)$ is a smooth function, $f_z(x)$ ($z \in [n] := \{1,2,\ldots,n\}$) is an individual loss associated with the z^{th} sample, $\|x\|_0$ denotes the l_0-norm of the parameter vector x which computes the number of nonzero entries in x (although the l_0 is not really a vector norm, we follow the convention here), and the integer k is the required sparsity parameter. Problem (1) plays an essential role in many statistical learning, machine learning, and signal processing problems and has been widely used in high-dimensional data analyses [11, 52, 5, 21]. The cardinality constraint, i.e., $\|x\|_0 \le k$, is nonconvex, so Problem (1) is a nonconvex constrained optimization problem and finding a global optimal solution x^* to Problem (1) is generally NP-hard [38].

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Table 1: Comparison of our approaches against the existing DP-GD-HT that is based on the regular gradient descent method. The DP-SGD-HT computes stochastic gradients based on mini-batches of size b which is $\ll n$ in practice, so it has less computational complexity than the DP-GD-HT (see the last column). A necessary assumption used to prove the convergence of the DP-SGD-HT is that the variance of stochastic gradients is upper bounded by σ_0^2. This assumption is no longer enforced when full gradients are used to correct for the variance of stochastic gradients as in the DP-SVRG-HT. The SCSG uses gradients computed on large data batches (size B) to correct for the variance of mini-batch-based gradients. The parameter $\psi \ll 1$ is defined in Theorem 6, so the DP-SCSG-HT also has smaller complexity than the DP-GD-HT.

Algorithm	Reference	Full Gradient	Constraint on variance σ_0^2	Computational Complexity
DP-GD-HT	[56]	Yes	No	$O(n\log(\frac{n^2\epsilon^2}{\log(1/\delta)}))$
DP-SGD-HT	This work	No	Yes	$O(b\log(\frac{n^2\epsilon^2}{\log(1/\delta)}))$
DP-SVRG-HT[1]	This work	Yes	No	$O(n\log(\frac{n^2\epsilon^2}{\log(1/\delta)}))$
DP-SCSG-HT	This work	No	Yes	$O(\min\{1,\psi\}\cdot n\log(\frac{n^2\epsilon^2}{\log(1/\delta)}))$

[1] A special case of DP-SCSG-HT, with batch size $B = n$.

1.1. Differential Privacy

In many sparse learning applications, sensitive information is present in training data, such as financial records, electronic medical records, or genomic data, which may trigger adversaries to attack the released model and to infer private information, via membership inference attack [49] or feature leakage [35]. To preserve user privacy, a learning algorithm/mechanism can be designed to satisfy the (ϵ,δ)-Differential Privacy (DP), which is a widely adopted mathematical definition of privacy-preserving and has become a standard in both academic fields and industry [16, 4, 39], due to its provable protection against adversaries. DP is formally defined by:

Definition 1 ((ϵ,δ)-DP [12])**.** *A randomized mechanism $\mathcal{M} : \mathcal{D} \to \mathcal{R}$ satisfies (ϵ,δ)-differential privacy ((ϵ,δ)-DP) if for any two adjacent datasets $D, D' \in \mathcal{D}$, for any output set $O \subseteq \mathcal{R}$, it holds that $\mathbb{P}[\mathcal{M}(D) \in O] \le e^{\epsilon}\cdot\mathbb{P}[\mathcal{M}(D') \in O]+\delta$, where $\mathcal{R}$ is the output space of $\mathcal{M}$ and the adjacent sets mean that D and D' differ by one entry.*

The (ϵ,δ)-DP implies that the mechanism $\mathcal{M}$ is ϵ-indistinguishable between two adjacent sets with probability $1-\delta$. For any output set O, $\frac{\mathbb{P}[\mathcal{M}(D)\in O]}{\mathbb{P}[\mathcal{M}(D')\in O]} \in [e^{-\epsilon}, e^{\epsilon}]$ with high probability $1-\delta$ and particularly, when ϵ is close to 0, $e^{\epsilon} \approx 1+\epsilon$, so $\frac{\mathbb{P}[\mathcal{M}(D)\in O]}{\mathbb{P}[\mathcal{M}(D')\in O]} \in [1-\epsilon, 1+\epsilon]$.

DP helps protect data privacy. For example, in the membership inference attack [49, 53] to a machine learning model that is published on cloud platforms, such as Amazon [9] and IBM [65], an attacker may be able to infer, based on the model prediction of an example X, whether X belongs to the training data on which the model has been trained. If the model is trained and provided by a cancer treatment center, and X is a patient of the center, his/her protected health information (i.e., being diagnosed with cancer) may be leaked if a non-DP algorithm, such as stochastic gradient descent, is used to train the machine learning model. However, if the model is trained using a DP algorithm, whether or not X belongs to the training data D (assuming the D' differs from D by just X), the model will only have limited variation (by e^{ϵ} as defined in Definition 1) for the attacker to detect [53].

The parameter ϵ is commonly referred to as privacy budget and δ is considered as the exceptional probability. In other words, with the probability δ, the model may vary beyond e^{ϵ} when training on adjacent training datasets. On the other hand, DP amounts to imposing a constraint to the model so the model learns from the training data as a whole but not much from an individual training example. Hence, DP often comes at a cost of losing prediction accuracy. A more restricted (smaller) privacy budget ϵ corresponds to worse prediction accuracy.

1.2. Optimization methods

For the unconstrained empirical risk minimization (ERM) problem of

$$\min_{x\in\mathbb{R}^d} f(x) := \frac{1}{n}\sum_{z=1}^{n} f_z(x), \tag{2}$$

the stochastic gradient descent (SGD) method [60] and its variants – variance reduced methods, have been extensively studied, including the stochastic variance reduced gradient (SVRG) [22, 47], stochastic average gradient (SAGA) [10], stochastic recursive gradient algorithm (SARAH) [41], stochastic path-integrated differential estimator (SPIDER) [17], and stochastically controlled stochastic gradient (SCSG) [27, 29] methods. However, these methods are proposed for unconstrained optimization and not directly suitable for solving Problem (1).

Problem (1) is difficult to solve even without the privacy-preserving consideration due to the non-convexity of the cardinality constraint. Existing research largely falls into the regimes of either matching pursuit methods [34, 44, 40, 18] or iterative hard thresholding (HT) methods [7, 20, 42]. Even though matching pursuit methods achieve remarkable success in quadratic loss functions (e.g., the l_0-constrained linear regression problems), they are required to find an optimal solution to min $f(x)$ on the identified support. The support is defined as the entries of x that are non-zero after hard thresholding. For an arbitrary loss (not quadratic), there is no analytical solution for this minimization problem which can then be time-consuming to solve [5]. Thus, iterative gradient-based HT methods have become popular for nonconvex sparse learning.

Iterative HT methods include the gradient descent HT (GD-HT) [20], stochastic gradient descent HT (SGD-HT) [42], hybrid stochastic gradient HT (HSG-HT) [64], stochastic variance reduced gradient HT (SVRG-HT) [32], and stochastically controlled stochastic gradient HT (SCSG-HT) [33] methods. These methods update the iterate x^t via gradient descent or its variants, and then apply the HT operator to enforce the sparsity of x^t. The computation can be concisely written as $x^{t+1} = \mathcal{H}_k(x^t - \eta v^t)$, where η is the learning rate, v^t can be the full gradient, stochastic gradient or variance reduced gradient at the t^{th} iteration, and $\mathcal{H}_k(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ denotes the HT operator that preserves the largest k elements of x in magnitude and sets other elements to 0. In a distributed computing setting, these iterative HT algorithms share gradients computed on a local device to other devices or a central server, and the shared gradients may leak private data when training a machine learning model.

1.3. The state of the art

In machine learning and deep learning communities, differentially-private algorithms have been extensively studied for unconstrained optimization problems including three approaches: output perturbation [61, 63], objective perturbation [8, 25], and gradient perturbation [6, 55, 1, 43, 62, 58]. However, privacy-preserving guarantee has been under-explored for sparse learning, especially in the stochastic optimization setting.

Several studies attempt to develop DP algorithms for sparse learning problems, such as the least absolute shrinkage and selection operator (Lasso) problem [25, 50, 51] or the cardinality constrained problem [56, 57]. The Lasso problem uses the l_1-norm as a convex surrogate of the l_0-norm and thus is a convex relaxation of Problem (1) , assuming a convex loss function f is used [25, 50, 51], which can be easily solved by gradient-based methods. However, the convex relaxation can result in large estimation bias to the solution of Problem (1) and has worse empirical performance [32]. Hence, researchers have also directly worked on the cardinality constrained problems and developed the DP-GD-HT algorithms [56, 57]. Although the DP-GD-HT algorithm has a solid utility analysis, it is not a stochastic algorithm, so it may not scale to large-scale problems. Because the DP-GD-HT method calculates the full gradient at each iteration, it is computationally expensive for high-dimensional problems with large sample size.

1.4. Contributions

In this paper, we propose and analyze two differentially private algorithms to solve Problem (1): DP-SGD-HT and DP-SCSG-HT. We provide privacy analysis of our algorithms based on the Rényi differential privacy (RDP). Moreover, we show their utility bounds, and computational complexities. We also experimentally verify our theoretical analysis for the proposed stochastic private methods, using real world sensitive financial records and medical records datasets. Our contributions are as follows,

- We design the first differentially private stochastic iterative HT method (DP-SGD-HT) that reduces the computational cost while guaranteeing the (ϵ, δ)-DP. Then in order to reduce the variance of stochastic gradients to further improve the learning accuracy, we develop another DP algorithm, called Stochastically Controlled Stochastic Gradient HT method (DP-SCSG-HT). However, the privacy analysis of DP-SCSG-HT is challenging since the number of iterations per epoch is a random variable. We provide a refined and sharp privacy loss estimation for DP-SCSG-HT based on RDP by controlling the effect of random iteration numbers.

- We perform convergence analysis for the proposed algorithms, and prove that the sequence $\{x_0, x_1, \cdots, x_T\}$ generated by either DP-SGD-HT or DP-SCSG-HT satisfies that $E[\|x_T - x^*\|^2] \leq \theta^T \|x_0 - x^*\|^2 + e$, where $0 < \theta < 1$ and e is the statistical bias due to the sparsity requirement and the injected Gaussian noise. It means that the two algorithms both enjoy a linear convergence rate with a linear factor θ under a statistical bias e, and hence match the results of non-stochastic DP-GD-HT which also converges in a linear rate.

- Given a privacy budget ϵ and an exception probability δ, we study the utility of the proposed algorithms. Despite the stochastic manner of the proposed algorithms, their utility is preserved. In other words, under the guarantee of (ϵ, δ)-DP, they incur an estimation error on the parameter x that is still upper bounded by $O(\frac{\log(1/\delta)}{n^2\epsilon^2})$, matching the utility bound obtained in [56] for the DP-GD-HT.

- We prove that the stochastic algorithms substantially reduce computational complexity from the DP-GD-HT as compared in Table 1. The complexity of the DP-SGD-HT linearly depends on $O(b\log(n))$ rather than $O(n\log(n))$ of the DP-GD-HT where b is the size of the mini-batch used to compute the stochastic gradients and can be much smaller than n. The computational complexity of the DP-SCSG-HT is $O(\min\{1, \psi\} \cdot n\log(n))$, where $\psi \ll 1$ in practice.

2. Preliminaries

We denote a vector by a lowercase letter, e.g. x, and the l_2-norm of a vector by $\|\cdot\|$. Let $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$ represent the asymptotic upper, lower, and tight bounds, respectively, and $E[\cdot]$ represent taking expectation over all random variables. We denote the integer set $\{1, ..., n\}$ by $[n]$, and $\nabla f(\cdot)$, $\nabla f_I(\cdot)$ and $\nabla f_z(\cdot)$ are the full gradient, stochastic gradient over a mini-batch $I \subset [n]$, and stochastic gradient over a training example indexed by $z \in [n]$, respectively. The symbol $\mathbb{I}(\cdot)$ is an indicator function, and $supp(x)$ means the support of x or the index set of non-zero elements in x. Let x^* be the optimal solution of Problem (1). The support $\mathcal{I}_{t+1}^{(j)} = supp(x^*) \cup supp(x_t^{(j)}) \cup supp(x_{t+1}^{(j)})$, is associated with the $(t+1)$-th iteration at the j-th epoch (and $\mathcal{I}$ is used throughout the paper without ambiguity); $\widetilde{\mathcal{I}} = supp\left(\mathcal{H}_{2k}\left(\nabla f\left(\boldsymbol{x}^*\right)\right)\right) \cup supp\left(\boldsymbol{x}^*\right)$. The projector $\pi_{\mathcal{I}}(x)$ gives a vector of the same length as x but zeros out the elements of x not indexed in $\mathcal{I}$.

Definition 2 (Rényi Divergence [48]). *Let P and Q be probability distributions on Ω. For $\alpha \in (1, \infty)$, the Rényi Divergence of order α between P and Q is defined as* $\mathrm{D}_\alpha(P\|Q) = \frac{1}{\alpha-1}\log\left(\int_\Omega P(x)^\alpha Q(x)^{1-\alpha}\mathrm{d}x\right)$.

Definition 3 ((α,ρ)-RDP [36]). *A randomized mechanism $\mathcal{M} : \mathcal{S} \to \mathcal{R}$ satisfies (α,ρ)-Rényi differential privacy (i.e., (α,ρ)-RDP) if for any two adjacent datasets $S, S' \in \mathcal{S}$, (i.e., differing by one example), where $\mathcal{S}$ is the space containing all possible sample sets that have n samples from an underlying distribution, the following inequality holds for $\alpha \in (1,\infty)$ and $\rho \in (0,\infty)$, $D_\alpha(\mathcal{M}(S)\|\mathcal{M}(S')) \leq \rho$, where $D_\alpha(\mathcal{M}(S)\|\mathcal{M}(S'))$ is the α-Rényi divergence between two distributions $\mathcal{M}(S)$ and $\mathcal{M}(S')$.*

Definition 4 (l_2-sensitivity [13]). *For any two adjacent datasets $S, S' \in \mathcal{S}$, the l_2-sensitivity $\Delta_2(q)$ of a query $q : \mathcal{S} \to \mathcal{R}$ is defined as $\Delta_2(q) = \sup_{S,S'} \|q(S) - q(S')\|_2$ where* sup *means taking the superior of the l_2-norm over all possible pairs of adjacent datasets.*

Remark 2.1. *In recent studies, the (α,ρ)-RDP has been used as an alternative of the (ϵ,δ)-DP. The (α,ρ)-RDP corresponds to the $(\rho + \frac{log(1/\delta)}{\alpha-1}, \delta)$-DP for any $\delta \in (0,1)$, which allows us to convert from (α,ρ)-RDP to (ϵ,δ)-DP.*

In machine learning, the widely used SGD algorithm subsamples mini-batches from the training dataset S. In a distributed computing environment, communicating among the distributed devices the stochastic gradients based on mini-batches rather than full gradients helps preserve data privacy. However, it imposes challenges to the traditional DP analysis based on the l_2-sensitivity, which is defined over the entire set S. Recently, the privacy amplification theorem for DP [23] shows that if $\mathcal{M}$ is (ϵ,δ)-DP, then $\mathcal{M}$ with the subsampling mechanism is $(O(\tau\epsilon), \tau\delta)$-DP where τ is the subsampling rate. We prefer to the (α,ρ)-RDP, because it has been proved that RDP has an analytical and tighter bound for subsampling mechanism as shown in Lemma 2.2 [59, 58].

Using the definition of the l_2-sensitivity, the following lemmas have been proved for the Gaussian mechanism and the composition rule of RDP.

Lemma 2.2 (Gaussian mechanism [58])**.** *Given a function $q : \mathcal{S} \to \mathcal{R}$, and $u \sim N(0, \sigma^2 I)$, the Gaussian mechanism $\mathcal{M} = q(S) + u$ satisfies $(\alpha, \frac{\alpha\Delta_2^2(q)}{2\sigma^2})$-RDP. If we apply $\mathcal{M}$ to subsamples that are uniformly sampled without replacement from S, $\mathcal{M}$ satisfies $(\alpha, \frac{5\tau^2\alpha\Delta_2^2(q)}{\sigma^2})$-RDP, if $\alpha \leq \log(\frac{1}{\tau(1+\sigma^2/\Delta_2^2(q))})$, where $\sigma^2 \geq 1.5\Delta_2^2(q)$ and τ is the subsampling rate.*

The privacy amplification theorem for RDP in [58] proves that the Gaussian perturbation parameter σ^2 needs to satisfy $\sigma^2 \geq 1.5\Delta_2^2(q)$ in order to derive an analytical formulation for ρ (in Lemma 2.2). It means that $\alpha \leq \log(\frac{1}{\tau(1+\sigma^2/\Delta_2^2(q))}) \leq -\log(2.5\tau)$. According to Definition 3, $\alpha > 1$ in the (α, ρ)-RDP, so it means that the sampling rate $\tau < e^{-1}/2.5 \approx 0.147$.

Lemma 2.3 (RDP composition [36])**.** *For two randomized mechanisms $\mathcal{M}_1 : \mathcal{S} \times \mathcal{R} \to \mathcal{R}$ and $\mathcal{M}_2 : \mathcal{S} \times \mathcal{R} \to \mathcal{R}$, if $\mathcal{M}_1$ satisfies (α, ρ_1)-RDP and $\mathcal{M}_2$ satisfies (α, ρ_2)-RDP, then the process of $\mathcal{M}_2(S, \mathcal{M}_1(S))$ (as a joint random process with $\mathcal{M}_1(S, \cdot)$) satisfies $(\alpha, \rho_1 + \rho_2)$-RDP.*

In this paper, a mechanism corresponds to a single SGD iteration with injected Gaussian noise. If our algorithm runs T iterations in total, we can recursively use Lemma 2.3 , so if the t-th mechanism satisfies (α, ρ_t)-RDP, then the composition of T mechanisms brings the entire algorithm to be $(\alpha, \sum_{t=1}^{T} \rho_t)$-RDP.

Lemma 2.4 (Invariant of post-processing [36])**.** *For mechanism $\mathcal{M}$ and post-processing mapping $g : \mathcal{R} \to \mathcal{R}$, if $\mathcal{M}$ satisfies (α, ρ)-RDP, then $g(\mathcal{M}(\cdot))$ is still (α, ρ)-RDP.*

Throughout the theoretical analyses, we assume that the objective function $f(x)$ in Eq.(1) satisfies the following assumptions that are commonly used in the study of nonconvex optimization:

Assumption 1. *Assume that the function $f_z(x)$ is l-Lipschitz continuous for any $z \in \{1, \cdots, n\}$. In other words, there exists a constant $l \geq 0$ such that $|f_z(x_1) - f_z(x_2)| \leq l\|x_1 - x_2\|, \forall x_1, x_2 \in \mathbb{R}^d, z \in [n]$.*

Remark 2.5. *For a differentiable function, the l-Lipschitz continuity implies that the gradient of the function is upper bounded, i.e., $\forall x$, $\|\nabla f_z(x)\| \leq l$. Assumption 1 is commonly used for deriving the l_2-sensitivity, such as in [55, 56]. In practice, instead of assuming the Lipschitz continuity of f_z, the gradient clipping technique in [1] can be used to ensure $\|\nabla f_z(x)\|$ is upper bounded by a pre-difined value l.*

Assumption 2. *Assume that the function $f(x)$ has σ_0^2-bounded stochastic gradient variance, i.e., $E[\|\nabla f_z(x) - \nabla f(x)\|^2] \leq \sigma_0^2, \forall x \in \mathbb{R}^d, z \in [n]$.*

For fair comparison with prior works on the HT methods [20, 42, 64, 32], we also use the same assumption as follows.

Assumption 3. *Assume that the function $f(x)$ is:*

(i) restricted ρ_s-strongly convex at the sparsity level s for a given $s \in \mathbb{N}_+$, i.e., there exists a constant $\rho_s > 0$ such that $\forall x_1, x_2 \in \mathbb{R}^d$ that $\|x_1 - x_2\|_0 \leq s$, we have $f(x_1) - f(x_2) - \langle \nabla f(x_2), x_1 - x_2 \rangle \geq \frac{\rho_s}{2}\|x_1 - x_2\|^2$;

(ii) restricted L_s smooth at the sparsity level s for a given $s \in \mathbb{N}_+$, i.e., there exists a constant $L_s > 0$ such that $\forall x_1, x_2 \in \mathbb{R}^d$ with $\|x_1 - x_2\|_0 \leq s$, we have $f(x_1) - f(x_2) - \langle \nabla f(x_2), x_1 - x_2 \rangle \leq \frac{L_s}{2}\|x_1 - x_2\|^2$.

To show that the HT operator is nearly non-expensive when k is much larger than optimal sparsity k^*, we have the following lemma.

Lemma 2.6 ([31])**.** *For $k > k^*$ and for any parameter $x \in \mathbb{R}^d$, we have $\|\mathcal{H}_k(x) - x^*\|_2^2 \leq (1 + \beta)\|x - x^*\|_2^2$,*

where $\beta = \frac{2\sqrt{k^*}}{\sqrt{k-k^*}}$ and $k^* = \|x^*\|_0$.

Our SCSG-based HT algorithm is essentially different with existing SVRG-based HT algorithms, where the number of iterations in inner loop is determined by a geometric distribution, which is formally defined as follows.

Definition 5 (Geometric Distribution)**.** *A random variable N follows a geometric distribution $Geom(\gamma)$, denoted as $N \sim Geom(\gamma)$, if N is a non-negative integer and the probability distribution is $P(N = k) = (1 - \gamma)\gamma^k, \quad \forall k = 0, 1, \cdots$ Then, we have the expectation of N, $E[N] = \frac{\gamma}{1-\gamma}$.*

In order to make the comparison of computational performance independent of the actual implementation of the algorithms, we use the number of IFO as defined below to measure computational complexity, which is a convention of stochastic optimization.

Definition 6 (IFO calls[2])**.** *An Incremental First-order Oracle (IFO) is a subroutine that takes a point $x \in \mathbb{R}^d$ and an index $z \in [n]$ and then returns a pair $(f_z(x), \nabla f_z(x))$.*

3. The DP-SGD-HT

To reduce computationally-expensive full gradients based differentially private hard thresholding algorithm, in this section, we propose the SGD-based hard thresholding algorithm, called DP-SGD-HT, as depicted in Algorithm 1, that can solve the sparsity constrained optimization problem Eq.(1) in a privacy-preserving manner.

Algorithm 1 DP-SGD-HT

Input: The maximal number of iterations T, initial state x^0, stepsize η, the mini-batch size $\{b_t\}$ at the t-th iteration, privacy parameters ϵ, δ and α
for $t = 1, 2, ..T$ **do**
  Sample uniformly a batch of examples, $I_t \subset \{1, ..., n\}$, where $|I_t| = b_t$
  $g_t = \nabla f_{I_t}(x_t)$
  $u_t \sim N(0, \sigma^2 \mathbf{I})$ where $\sigma^2 = \frac{40\alpha l^2 T}{n^2 \epsilon}$
  $x_{t+1} = \mathcal{H}_k(x_t - \eta(g_t + u_t))$
end for

At the core of Algorithm 1 is a stochastic gradient perturbation procedure at each iteration. Specifically, we perturb the stochastic gradient in an iteration with Gaussian noise $N(0, \sigma^2 \mathbf{I})$, instead of perturbing computationally-expensive full gradients used in DP-GD-HT algorithms [56, 57, 54]. We then make use of the composition rule and privacy-amplification by subsampling of differential privacy to prove an upper bound on the total privacy loss. Note that the DP-SGD-HT consists of the original SGD-HT as a special case if the noise variance $\sigma^2 = 0$ although we provide a suggested value of σ^2 in Algorithm 1. In the following, we provide the privacy analysis, convergence guarantee, and utility bound of our proposed DP-SGD-HT algorithm.

3.1. Differential Privacy Guarantee of the DP-SGD-HT

We show that Algorithm 1 satisfies the DP. Specifically, we prove that it satisfies the (α, ρ)-RDP, and then we convert it to the format of (ϵ, δ)-DP as discussed in Remark 2.1 to compare with existing results.

Theorem 3.1. *Algorithm 1 satisfies the (ϵ, δ)-DP, with $b_t = b$, and $\sigma^2 = \frac{40\alpha l^2 T}{n^2 \epsilon}$, where $\alpha = 1 + \frac{2\log(1/\delta)}{\epsilon}$, if $\alpha \leq \log(\frac{n^3 \epsilon}{ln^2 b\epsilon + 10 l\alpha T b^3})$ and $\frac{10 b^2 \alpha T}{n^2 \epsilon} \geq 1.5$.*

Proof. At the $(t+1)$-th iteration of Algorithm 1, we have the update rule: $x_{t+1} = \mathcal{H}_k(x_t - \eta(g_t + u_t))$, where $g_t = \nabla f_{I_t}(x_t)$ and $u_t \sim N(0, \sigma^2 I)$.

We consider the following query function on a set S of n training examples, $q_t(S) = \frac{1}{b_t}\sum_{i=1}^n \nabla f_i(x_t)$. For any two adjacent datasets S and S', let us index the different examples in S and S' by z and z'. By Definition 4 and Remark 2.5, the l_2-sensitivity $\Delta_2(q_t)$ of q_t is:

$$\Delta_2(q_t) = \sup_{S,S'} \|q_t(S) - q_t(S')\| = \sup_{z,z'} \|\frac{1}{b_t}\nabla f_z(x_t) - \frac{1}{b_t}\nabla f_{z'}(x_t)\| \leq \frac{2l}{b_t}.$$

By Lemma 2.2, the Gaussian mechanism $\mathcal{M} = q_t(S) + u_t$ is $(\alpha, \frac{2\alpha l^2}{b_t^2 \sigma^2})$-RDP for the query function $q_t(S)$. We now consider $q_t(S)$ calculated on a subsample I_t that is uniformly drawn from S, $\tilde{q}_t(S) = \frac{1}{b_t}\sum_{z \in I_t} \nabla f_z(x_t)$. Because the sampling rate $\tau = \frac{b_t}{n}$, substituting the formula of τ and $\Delta_2(q_t)$ into Lemma 2.2 yields that $\tilde{\mathcal{M}} = \tilde{q}_t(S) + u_t$ is

$(\alpha, \frac{20\alpha l^2}{n^2\sigma^2})$-RDP, if $\alpha \le \log\left(\frac{n}{b_t(1+\frac{\sigma^2 b_t^2}{4l^2})}\right)$ and $\sigma^2 \ge \frac{6l^2}{b_t^2}$. Due to the invariant property of post-processing of RDP [36], we know that the mechanism $\tilde{\mathcal{M}}' = \mathcal{H}_k(x_t - \eta\tilde{\mathcal{M}})$ is $(\alpha, \frac{20\alpha l^2}{n^2\sigma^2})$-RDP. By Lemma 2.3, after running T iterations, we obtain that Algorithm 1 satisfies the $(\alpha, \frac{20\alpha l^2 T}{n^2\sigma^2})$-RDP, and correspondingly $(\frac{20\alpha l^2 T}{n^2\sigma^2} + \frac{\log(1/\delta)}{\alpha-1}, \delta)$-DP for $\delta \in (0, 1)$ according to Remark 2.1. Let

$$\frac{20\alpha l^2 T}{n^2\sigma^2} + \frac{\log(1/\delta)}{\alpha - 1} = \epsilon,$$

and $\alpha = 1 + \frac{2\log(1/\delta)}{\epsilon}$, which implies that $\sigma^2 = \frac{40\alpha l^2 T}{n^2\epsilon}$. This σ^2 formula gives us the suggested value for the injected Gaussian noise.

Therefore, Algorithm 1 satisfies $(\epsilon, \delta)-$DP if we use $b_t = b$, $\alpha = 1 + \frac{2\log(1/\delta)}{\epsilon}$, $\sigma^2 = \frac{40\alpha l^2 T}{n^2\epsilon}$ in Algorithm 1, and if $\alpha \le \log(\frac{n^3\epsilon}{ln^2 b\epsilon + 10l\alpha T b^3})$ and $\frac{10b^2\alpha T}{n^2\epsilon} \ge 1.5$. □

Theorem 3.1 guarantees that the DP-SGD-HT algorithm is $(\epsilon, \delta)-$DP, and derives an analytical formula for σ^2, the parameter of the added Gaussian noise. Because of the subsampling technique used in Algorithm 1, a constraint on α is introduced. This constraint is similar to the constraint introduced in [1] for deep learning applications with the moments accountant technique, while our α has a closed-form solution. If we directly work on (ϵ, δ)-DP and apply the strong composition theorem in [14], such a constraint can be removed, but an extra $\log(T/\delta)$ factor will be introduced to σ^2 and hence will worsen the utility bound derived in a later section.

3.2. Convergence Guarantee of the DP-SGD-HT

In order to make the SGD-HT satisfy the (ϵ, δ)-DP, Algorithm 1 has included a randomized Gaussian process, which may affect the convergence of the original SGD-HT method, and may alter the convergence rate. We examine the convergence of the DP-SGD-HT by developing an upper bound on the distance between the estimator x_t and the optimal x^*, i.e. $E[\|x_t - x^*\|^2]$ in Theorem 3.2.

Theorem 3.2. *Suppose that $f(x)$ satisfies Assumptions 1 and 3, $k^* = \|x^*\|_0$, $k \ge 4k^*(12\kappa_s - 1)^2 + k^*$ where $\kappa_s = \frac{L_s}{\rho_s}$ is the condition number of $f(x)$. Define $\tilde{\mathcal{I}} = supp(x^*) \cup supp(\mathcal{H}_{2k}(\nabla f(x^*)))$, and let $\eta = \frac{1}{6L_s}$. If the variance of stochastic gradients $\sigma_0^2 \le kb_t\sigma^2$, then we can get*

$$E[\|x_t - x^*\|^2] \le \theta_1^t \|x_0 - x^*\|^2 + \frac{1}{1-\theta_1}\frac{1+\beta}{12L_s^2}\left\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\right\|^2 + \frac{1}{1-\theta_1}\frac{k(1+\beta)}{6L_s^2}\sigma^2, \tag{3}$$

where $\beta = \frac{2\sqrt{k^*}}{\sqrt{k-k^*}}$, $\theta_1 = (1 + \frac{2\sqrt{k^*}}{\sqrt{k-k^*}})(1 - \frac{1}{12\kappa_s}) < 1$.

Proof. Assume that $y_t = x_t - \eta(\pi_{\mathcal{I}}(g_t + u_t))$, then

$$\begin{aligned}
E[\,\|y_t - x^*\|^2] &= E[\|x_t - \eta(\pi_{\mathcal{I}}(g_t + u_t)) - x^*\|^2] \\
&\overset{①}{=} E[\|x_t - x^*\|^2] + \eta^2 E[\|\pi_{\mathcal{I}}(g_t)\|^2] + \eta^2 E[\|\pi_{\mathcal{I}}(u_t)\|^2] - 2\eta E[\langle x_t - x^*, \pi_{\mathcal{I}}(g_t)\rangle] \\
&\overset{②}{\le} E[\|x_t - x^*\|^2] + \eta^2 E[\|\pi_{\mathcal{I}}(g_t)\|^2] + \eta^2 E[\|\pi_{\mathcal{I}}(u_t)\|^2] - 2\eta E[f(x_t) - f(x^*)] \\
&\overset{③}{\le} E[\|x_t - x^*\|^2] + 2\eta(3\eta L_s - 1)E[f(x_t) - f(x^*)] + 6\eta^2 L_s E[\langle \pi_{\mathcal{I}}(\nabla f(x^*)), x_t - x^*\rangle] \\
&\quad + \frac{3\eta^2}{b_t}\sigma_0^2 + 3\eta^2 E[\|\pi_{\mathcal{I}}(\nabla f(x^*))\|^2] + \eta^2 E[\|\pi_{\mathcal{I}}(u_t)\|^2],
\end{aligned}$$

where ① holds because u_t is independent of all other random variables, such as x_t, and $E[u_t] = 0$; ② holds because $E[\langle x_t - x^*, \pi_{\mathcal{I}}(g_t)\rangle] \ge E[f(x_t) - f(x^*)]$, which is derived from restricted strong convexity, ③ holds by Lemma 4 in [64] that

$$E[\|\pi_{\mathcal{I}}(g_t)\|^2] \le 6L_s E[f(x_t) - f(x^*)] + 6L_s E[\langle \pi_{\mathcal{I}}(\nabla f(x^*)), x_t - x^*\rangle] + \frac{3}{b_t}\sigma_0^2 + 3\,\|\pi_{\mathcal{I}}(\nabla f(x^*))\|^2\,,$$

and σ_0^2 is finite as in Assumption 2.

By the restricted ρ_s-strong convexity and setting $\eta \le \frac{1}{3L_s}$ yields

$$E[\|y_t - x^*\|^2] \le (1 + \rho_s\eta(3\eta L_s - 1))E[\|x_t - x^*\|^2] + 2\eta(6\eta L_s - 1)E[\langle \nabla_{\mathcal{I}} f(x^*), x_t - x^*\rangle] + \frac{3\eta^2}{b_t}\sigma_0^2 + 3\eta^2 E[\|\pi_{\mathcal{I}}(\nabla f(x^*))\|^2] + \eta^2 E[\|\pi_{\mathcal{I}}(u_t)\|^2].$$

Here the operator $\pi_I(x)$, as defined in Section II, zeros out the elements of x not indexed in $\mathcal{I}$. Because the size of support $\mathcal{I}$ is $3k$ and $u_t \sim N(0, \sigma^2 I)$, we have $E[\|\pi_I(u_t)\|^2] \le 3k\sigma^2$. Then if $\eta = \frac{1}{6L_s}$, we get

$$E[\|y_t - x^*\|^2] \le (1 - \frac{1}{12\kappa_s})E[\|x_t - x^*\|^2] + \frac{1}{12L_s^2}E[\|\pi_{\mathcal{I}}(\nabla f(x^*))\|^2] + \frac{1}{b_t}\frac{1}{12L_s^2}\sigma_0^2 + \frac{k}{12L_s^2}\sigma^2.$$

By Lemma 2.6, we can obtain

$$\begin{aligned} E[\|x_{t+1} - x^*\|^2] &\le (1 + \frac{2\sqrt{k^*}}{\sqrt{k - k^*}})E[\|y_t - x^*\|^2] \\ &= \theta_1 E\|x_t - x^*\|^2 + \frac{1+\beta}{12L_s^2}E[\|\pi_{\mathcal{I}}(\nabla f(x^*))\|^2] + \frac{1+\beta}{12L_s^2 b_t}\sigma_0^2 + \frac{k(1+\beta)}{12L_s^2}\sigma^2, \end{aligned}$$

where $\theta_1 = (1 + \frac{2\sqrt{k^*}}{\sqrt{k-k^*}})(1 - \frac{1}{12\kappa_s})$ and $\beta = \frac{2\sqrt{k^*}}{\sqrt{k-k^*}}$. If we further require $\theta_1 = (1 + \frac{2\sqrt{k^*}}{\sqrt{k-k^*}})(1 - \frac{1}{12\kappa_s}) < 1$, and $\frac{1}{b_t}\frac{1+\beta}{12L_s^2}\sigma_0^2 \le \frac{k(1+\beta)}{12L_s^2}\sigma^2$ i.e. $k \ge 4k^*(12\kappa_s - 1)^2 + k^*$, and $\sigma_0^2 \le kb_t\sigma^2$, we get

$$E[\|x_T - x^*\|^2] \le \theta_1^T \|x_0 - x^*\|^2 + \frac{1}{1-\theta_1}\frac{1+\beta}{12L_s^2}\left\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\right\|^2 + \frac{1}{1-\theta_1}\frac{k(1+\beta)}{6L_s^2}\sigma^2.$$

$\square$

Theorem 3.2 shows that the DP-SGD-HT converges to x^* with an estimation error bias in a linear convergence rate and the convergence factor is specified by θ_1. This result matches that of the non-DP SGD-HT [42] and HSGD-HT [64], because both of them also achieve a linear convergence rate. Precisely, the estimation error is upper bounded by the sum of three terms in Eq.(3). The first term approaches 0 when the number of iterations t goes to infinity. The second term is a statistical bias term due to the sparsity constraint on the solution x^*. If x^* is sufficiently close to the unconstrained minimizer of f when k is chosen to be large, then $\|\nabla f(x^*)\|$ becomes close to 0. The last term is another bias term generated by the perturbation noise of the Gaussian mechanism that guarantees differential privacy. When this noise specified by σ^2 approaches 0, the third term vanishes. The second and third terms together form an estimation error floor that does not vanish with increasing iterations. Compared with the original SGD-HT algorithm [42], the upper bound Eq.(3) incurs an additional term determined by σ^2. However, our analysis no longer requires that the condition number $\kappa_s \le \frac{4}{3}$ in [42]) which is difficult to satisfy.

3.3. *The Utility Bound of the DP-SGD-HT*

It is important to examine if the inclusion of a randomized process to the update rule affects the utility of the optimization algorithm. With the above convergence analysis, the utility of Algorithm 1 is reserved, which is characterized by the following theorem.

Theorem 3.3 (Utility)**.** *Under the same setting of Theorem 3.2, if we let $T = O(\log(\frac{n^2\epsilon^2}{\log(1/\delta)}))$, the output of Algorithm 1, x_T, satisfies*

$$E[\|x_T - x^*\|^2] \le \frac{1}{1-\theta_1}\frac{(1+\beta)}{12L_s^2}\left\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\right\|^2 + O(\frac{\log(1/\delta)}{n^2\epsilon^2}\log(\frac{n^2\epsilon^2}{\log(1/\delta)})). \tag{4}$$

Here the expectation is taken over all the randomness of the algorithm, including both the subsampling for computing stochastic gradients and the random noise added for ensuring differential privacy.

Proof. Because in Eq.(3), the third term is determined by the noise level σ^2, when the first term is less than the third term, having more iterations may not improve the bound further. (Note that the second term is due to the sparsity of the solution which is not an amenable algorithm parameter.)

Setting $\theta_1^T \|x_0 - x^*\|^2 \leq \frac{1}{1-\theta_1} \frac{k(1+\beta)}{6L_s^2} \frac{40\alpha l^2}{n^2\epsilon}$ yields

$$T = \log_{\theta_1}\left(\frac{1}{\|x_0 - x^*\|^2} \frac{1}{1-\theta_1} \frac{k(1+\beta)}{6L_s^2} \frac{40\alpha l^2}{n^2\epsilon}\right) = O(\log(\frac{n^2\epsilon^2}{\log(1/\delta)})).$$

Hence, we get the upper bound of $E[\|x_T - x^*\|^2]$. □

Remark 3.4. *Theorem 3.3 implies that the DP-SGD-HT approximates a sparse optimal solution with an upper bound of* $O(\left\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\right\|^2 + \frac{\log(1/\delta)}{n^2\epsilon^2})$. *The term* $O(\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\|^2)$ *specifies the sparsity-induced statistical error, which approaches* 0 *if* x^* *is sufficiently close to an unconstrained minimizer of* $f(x)$*, so it represents the sparsity-induced bias to the solution of the unconstrained optimization problem. The second term* $O(\frac{\log(1/\delta)}{n^2\epsilon^2})$ *is induced by the Gaussian mechanism and will be large with small* ϵ *and* δ*, which is corresponding to the high privacy guarantee situation, and hence plays the dominating role in high privacy regime.*

Based on the convergence analysis, we can further analyze the computational complexity of the DP-SGD-HT, which specifies an upper bound on the total number of IFOs that Algorithm 1 needs to calculate during the training process in Corollary 3.4.1.

Corollary 3.4.1 (Computational Complexity)**.** *Under the same conditions of Theorem 3.3, the number of IFO calls is* $T \times b = O(b\log(\frac{n^2\epsilon^2}{\log(1/\delta)}))$.

Note that early analysis of the DP-GD-HT shows that the computational complexity of the non-stochastic version is in the order of $O(n\log(n))$ [56]. Our stochastic version with a computational complexity of $O(b\log(n))$ is better because the size of mini-batch b is generally much smaller than the training sample size n.

4. The DP-SCSG-HT

Although our proposed DP-SGD-HT can significantly reduce the computational cost of full gradients algorithms, the randomness of batch data sampling introduces additional variance to gradient estimation. Leveraging the variance reduction techniques, we propose the DP-SCSG-HT algorithm, which effectively enhances and accelerates the convergence and utility of DP-SGD-HT algorithm. In particular, the variance of stochastic gradients can be well controlled by full or large-batch gradients calculated at each snapshot in a variance reduction technique. Because computation may be wasted if full gradients are calculated as discussed in [19], we calculate a batch gradient to correct the mini-batch stochastic gradients once in several iterations. We use the stochastically controlled stochastic gradient method, so the number of iterations in the inner loop is determined by a geometric distribution.

As shown in Algorithm 2, the DP-SCSG-HT has two loops: the outer loop (Lines 2 - 16) and the inner loop (Lines 9 - 14). A batch gradient is computed at each outer iteration (Line 5) to approximate the full gradient so the batch size B is set to be large. In an inner loop, stochastic gradients are calculated on mini-batches, which have a much smaller size b. Note that different from DP-SGD-HT algorithm, in DP-SCSG-HT, the number of iterations in the inner loop $N^{(j)}$ requires to be determined, which we suggest two options: in option I, $N^{(j)}$ is randomly drawn from a geometric distribution, similar to the methods in [28, 3, 15]; in option II a deterministic constant $\frac{B}{b}$ is used and $\frac{B}{b}$ is the expectation of the geometric distribution Geom $(B/(B+b))$. In practice, both options are applicable, and as observed in [28, 15], option II can be more stable, because setting $N^{(j)}$ to a constant eliminates the variance of $N^{(j)}$ introduced in option I. However, with option I, the property of geometric distribution makes the theoretical analysis of Algorithm 2 more concise. Thus, we perform the theoretical analysis of the DP-SCSG-HT method based on both of the options, which provides a more general setting for both theoretical analysis and practical applications.

Algorithm 2 DP-SCSG-HT

Input: The maximal number of outer loops $\mathcal{J}$, initial state $\tilde{x}^1$, stepsize η, batch sizes B, and b, σ_1, and σ_2
for $j = 1, 2, ..\mathcal{J}$ **do**
  Randomly pick $I^{(j)} \subset \{1, ..., n\}$, where $|I^{(j)}| = B$
  $u_{t,1}^{(j)} \sim N(0, \sigma_1^2 \mathbf{I})$
  $\tilde{\mu}^{(j)} = \nabla f_{I^{(j)}}(\tilde{x}^{(j)}) + u_{t,1}^{(j)}$
  $x_0^{(j)} = \tilde{x}^{(j)}$
  option I: Generate $N^{(j)} \sim \text{Geom}\ (B/(B+b))$
  option II: $N^{(j)} = \frac{B}{b}$
  for $t = 1, 2, \dots, N^{(j)}$ **do**
    Randomly pick $I_t^{(j)} \subset \{1, ..., n\}$, where $|I_t^{(j)}| = b$
    $u_{t,2}^{(j)} \sim N(0, \sigma_2^2 \mathbf{I})$
    $v_t^{(j)} = \nabla f_{I_t^{(j)}}(x_t^{(j)}) - \nabla f_{I_t^{(j)}}(\tilde{x}^{(j)}) + \tilde{\mu}^{(j)} + u_{t,2}^{(j)}$
    $x_t^{(j)} = \mathcal{H}_k(x_{t-1}^{(j)} - \eta v_t^{(j)})$
  end for
  set $\tilde{x}^{j+1} = x_{N^{(j)}}^{(j)}$
end for

4.1. Differential Privacy Guarantee of the DP-SCSG-HT

DP analysis can be difficult for DP-SCSG-HT, because the number of inner iterations N is a random variable and we have to bound N based on the property of geometric distribution. We first show that the proposed DP-SCSG-HT algorithm satisfies the (α, ρ)-RDP, which is then converted into the (ϵ, δ)-DP as summarized in Theorem 4.1. Different from the analysis of the DP-SGD-HT, every updating iteration based on the stochastic variance reduced gradient deals with two different subsampling: $I^{(j)}$ at a snapshot and $I_t^{(j)}$ at each iteration of the inner loop. A proof sketch is provided below and more details are given in the Appendix.

Theorem 4.1. *Let the maximal number of epochs be $\mathcal{J}$, and $\frac{\sigma_1^2}{160} = \frac{\sigma_2^2}{40} = \sigma^2$ where $\sigma^2 = \frac{2CBl^2\alpha\mathcal{J}}{bn^2\epsilon}$ for a constant $C > 0$, and $\alpha = 1 + \frac{2\log(2/\delta)}{\epsilon}$. Algorithm 2 satisfies the (ϵ, δ)–DP if $\alpha \leq \log(\frac{bn^3\epsilon}{Bbn^2\epsilon + 20CB^4\alpha\mathcal{J}})$, $\frac{20\alpha CBb\mathcal{J}}{n^2\epsilon} \geq 1.5$ and $1 - (1 - \frac{\delta}{2})^{\frac{1}{\mathcal{J}}} \geq e^{-(C-1-\ln(C))}$.*

Proof Sketch: Let S be a set of n training examples. We consider the following two queries:

$$\tilde{q}_{t,1}^{(j)}(S) = \nabla f_{I^{(j)}}(\tilde{x}^{(j)}),$$

$$\tilde{q}_{t,2}^{(j)}(S) = \nabla f_{I_t^{(j)}}(x_{t-1}^{(j)}) - \nabla f_{I_t^{(j)}}(\tilde{x}^{(j)}) + \tilde{\mu}^{(j)},$$

given $\tilde{\mu}^{(j)}$.

Part I. For $\tilde{q}_{t,1}^{(j)}(S)$, we consider the following query function: $q_{t,1}^{(j)}(S) = \frac{1}{B}\sum_{z=1}^{n} \nabla f_z(\tilde{x}^{(j)})$. By Lemma 2.2, for query function $q_{t,1}^{(j)}(S)$, the Gaussian mechanism $\mathcal{M}_1 = q_{t,1}^{(j)}(S) + u_{t,1}^{(j)}$, where $u_{t,1}^{(j)} \sim N(0, \sigma_1^2 I)$ is $(\alpha, \frac{\alpha\Delta_2^2(q_{t,1}^{(j)})}{2\sigma_1^2})$-RDP, and precisely is $(\alpha, \frac{4\alpha l^2}{B^2\sigma_1^2})$-RDP.

Then, let us examine the subsampling query $\tilde{q}_{t,1}^{(j)}(S)$. The mechanism $\tilde{\mathcal{M}_1}^{(j)} = \tilde{q}_{t,1}^{(j)}(S) + u_{t,1}^{(j)}$ is $(\alpha, \frac{20\alpha l^2}{n^2\sigma_1^2})$-RDP, if $\alpha \leq \log(\frac{n}{B(1+\sigma_1^2 B^2/4l^2)})$ and $\frac{B^2\sigma_1^2}{4l^2} \geq 1.5$.

Part II. For $\tilde{q}_{t,2}^{(j)}(S)$, we first examine the following query function: $q_{t,2}^{(j)}(S) = \frac{1}{b}\sum_{z=1}^{n} \nabla f_z(x_{t-1}^{(j)}) - \frac{1}{b}\sum_{z=1}^{n} \nabla f_z(\tilde{x}^{(j)}) + \tilde{\mu}^{(j)}$, conditioning on $\tilde{\mu}^{(j)}$. By Lemma 2.2, for the query function $q_{t,2}^{(j)}(S)$, the Gaussian mechanism $\mathcal{M}_2 = q_{t,2}^{(j)}(S) + u_{t,2}^{(j)}$, where $u_{t,2}^{(j)} \sim N(0, \sigma_2^2 I)$ is $(\alpha, \frac{\alpha\Delta_2^2(q_{t,2}^{(j)})}{2\sigma_2^2})$-RDP.

Then, we examine the following query with the subsample $I_t^{(j)}$, $\tilde{q}_{t,2}^{(j)}(S) = \nabla f_{I_t^{(j)}}(x_{t-1}^{(j)}) - \nabla f_{I_t^{(j)}}(\tilde{x}^{(j)}) + \tilde{\mu}^{(j)}$ conditioning on $\tilde{\mu}^{(j)}$. The mechanism $\tilde{\mathcal{M}}_2 = \tilde{q}_{t,2}^{(j)}(S) + u_{t,2}^{(j)}$ is $(\alpha, \frac{80\alpha l^2}{n^2\sigma_2^2})$-RDP, if $\alpha \le \log(\frac{16nl^2}{16l^2b+\sigma_2^2b^3})$ and $\frac{b^2\sigma_1^2}{16l^2} \ge 1.5$.

Combining the analyses of **Part I** and **Part II**, and setting $\frac{\sigma_1^2}{160} = \frac{\sigma_2^2}{40} = \sigma^2$, yield that $(\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2)$ satisfies $(\alpha, \frac{l^2\alpha}{n^2\sigma^2})$-RDP, by the composition rule in Lemma 2.3.

Because $N^{(j)} \sim Geom(B/(B+b))$ is a random variable, we need to bound $N^{(j)}$ in order to apply the composition rule. Hence, we consider the event $\mathbb{E} = \{N^{(j)} \le \frac{CB}{b}$ for $1 \le j \le \mathcal{J}\}$ with the probability of $\mathbb{E}$ as $\mathbb{P}(\mathbb{E})$. We prove that there exists a constant C satisfying $e^{-(C-1-\ln(C))} \le 1 - (1-\frac{\delta}{2})^{\frac{1}{\mathcal{J}}}$, such that the number of $N^{(j)}$ is upper bounded by $\frac{CB}{b}$ with at least the probability $(1-\frac{\delta}{2})^{\frac{1}{\mathcal{J}}}$. Hence, $\mathbb{P}(\mathbb{E}) = \Pi_{j=1}^{\mathcal{J}} P(\mathbb{E}_j) \ge 1 - \frac{\delta}{2}$ where $\mathbb{E}_j$ is the event of $N^{(j)} \le \frac{CB}{b}$ for $\forall j$. Conditioning on event $\mathbb{E}$, we can show that Algorithm 2 satisfies the $(\frac{CB\alpha l^2\mathcal{J}}{bn^2\sigma^2} + \frac{\log(2/\delta)}{\alpha-1}, \delta/2)$-DP, which is the $(\epsilon, \delta/2)$-DP if $\alpha = 1 + \frac{2\log(2/\delta)}{\epsilon}$ and $\sigma^2 = \frac{2CB\alpha l^2\mathcal{J}}{bn^2\epsilon}$. By Definition 1, for adjacent datasets S, S' and any output O, we obtain $\mathbb{P}[\mathcal{M}(S) \in O|\mathbb{E}] \le e^{\epsilon} \cdot \mathbb{P}[\mathcal{M}(S') \in O|\mathbb{E}] + \delta/2$. Therefore, we further obtain

$$
\begin{aligned}
&\mathbb{P}[\mathcal{M}(S) \in O] \\
&= \mathbb{P}[\mathcal{M}(S) \in O|\mathbb{E}] \cdot \mathbb{P}(\mathbb{E}) + \mathbb{P}[\mathcal{M}(S) \in O|\mathbb{E}^c] \cdot \mathbb{P}(\mathbb{E}^c) \\
&\le (e^{\epsilon} \cdot \mathbb{P}\left[\mathcal{M}(S') \in O \middle| \mathbb{E}\right] + \delta/2)\mathbb{P}(\mathbb{E}) + \delta/2 \\
&\le e^{\epsilon} \cdot \mathbb{P}\left[\mathcal{M}(S') \in O \middle| \mathbb{E}\right] \cdot \mathbb{P}(\mathbb{E}) + \delta \\
&\le e^{\epsilon} \cdot \mathbb{P}\left[\mathcal{M}(S') \in O\right] + \delta,
\end{aligned}
$$

where $\mathbb{E}^c$ is the complementary event of $\mathbb{E}$. Therefore, Algorithm 2 satisfies the (ϵ, δ)-DP. □

For option II, analysis becomes easier because the number of iterations in an epoch ($N^{(j)}$) is fixed and the composition rule for RDP can be directly applied. We can easily show that the DP-SCSG-HT with option II also satisfies the DP with a constant $C = 1$, $\alpha = 1 + \frac{2\log(1/\delta)}{\epsilon}$.

Remark 4.2. *The variance of the injected Gaussian noise is required to be $\sigma^2 = \frac{40\alpha l^2 T}{n^2\epsilon}$ for the DP-SGD-HT where T is the total number of iterations. Compared with the DP-SGD-HT, the variances of Gaussian noises in the DP-SCSG-HT σ_1^2 and σ_2^2 satisfy $\frac{\sigma_1^2}{160} = \frac{\sigma_2^2}{40} = \sigma^2$, and the value of $\sigma^2 = \frac{2CBl^2\alpha\mathcal{J}}{bn^2\epsilon}$ can be much smaller. If $C = 1$ for option II, the number of total inner iterations for the DP-SCSG-HT is $\frac{B\mathcal{J}}{b}$, which is usually smaller than T in practice for large-scale problems. Therefore, practically, DP-SCSG-HT achieves better estimation, due to the lower bias derived from the lower perturbation noise, as analyzed in Theorem 4.2 and empirically observed in the experiments section.*

4.2. Convergence Guarantee of the DP-SCSG-HT

We examine how adding Gaussian noises in Algorithm 2 to preserve data privacy can alter the convergence of the algorithm. We develop an upper bound on the distance between the estimator x_t and the optimal x^*.

Theorem 4.3. *Suppose that $f(x)$ satisfies Assumptions 1 and 3. Define $\tilde{\mathcal{I}} = supp(x^*) \cup supp(\mathcal{H}_{2k}(\nabla f(x^*)))$. Let $k^* = \|x^*\|_0$, the restricted condition number of $f(x)$, $\kappa_s = \frac{L_s}{\rho_s} \ge 1$, and $\beta = \frac{2\sqrt{k^*}}{\sqrt{k-k^*}} \le min\{\frac{b}{B}, \frac{1}{64\kappa_s^2-1}\}$. If the variance of stochastic gradients $\mathbb{I}(B<n)\sigma_0^2 \le kB\sigma^2$, then we can get,*

$$
E[\|\tilde{x}^{(j+1)} - x^*\|^2] \le \theta_2^{j+1}\|\tilde{x}^{(0)} - x^*\|^2 + \frac{1}{128(1-\theta_2)\gamma L_s^2\kappa_s^2}\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\|^2 + \frac{7k}{1024(1-\theta_2)\gamma L_s^2\kappa_s^2}\sigma^2, \quad (5)
$$

where $\theta_2 = 1 - \frac{1}{\frac{64\kappa_s^2}{1+\beta}\frac{b}{B} + \frac{13}{8}} < 1$, $\gamma = \frac{\frac{b}{B}-\beta}{1+\beta} + \frac{14\kappa_s - 1}{512\kappa_s^3} > 0$ and $\mathbb{I}(\cdot)$ is an indicator function.

Proof sketch: We first give some preparations and then show the line of main proof.
1) Preparations. In our analysis, we introduce an error term $e^{(j)} = \nabla f_{I^{(j)}}(\tilde{x}^{(j)}) - \nabla f(\tilde{x}^{(j)})$, which plays an important role in the flow of the derivation, and is one of the major differences from the analysis of the existing SVRG-HT [32]. Because $v_t^{(j)} = \nabla f_{I_t^{(j)}}(x_t^{(j)}) - \nabla f_{I_t^{(j)}}(\tilde{x}^{(j)}) + \tilde{\mu}^{(j)} + u_{t,2}^{(j)}$ is the updating direction at the t^{th} iteration of the j^{th} epoch in

Algorithm 2, $e^{(j)}$ is the bias of the updating direction $v_t^{(j)}$, where $E_{I_t^{(j)}}[v_t^{(j)}] = \nabla f(x_t^{(j)}) + e^{(j)}$ and $E_{I_t^{(j)}}$ is the expectation over stochastic sampling $I_t^{(j)}$. We show that the variance of the term $e^{(j)}$ can be bounded as

$$E[\|\pi_{\mathcal{I}}(e^{(j)})\|^2] \leq 2L_s^2 \frac{\mathbb{I}(B<n)}{B} E[\|\tilde{x}^{(j)} - x^*\|^2] + 2\frac{\mathbb{I}(B<n)}{B}\sigma_0^2, \tag{6}$$

which will diminish to zero with an increasing batch size B. The above bound gives extra flexibility to adaptively adjust the batch size B based on the variance of Gaussian perturbed noise.

Before diving into the detailed proof, we also need to analyze the term for the variance of stochastic gradient direction - $E_{I_t^{(j)}}[\|\pi_{\mathcal{I}}(v_t^{(j)})\|^2]$ on $\pi_{\mathcal{I}}(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, which is a projection operator to support $\mathcal{I}$. Then we have:

$$\begin{aligned} E_{I_t^{(j)}}[\|\pi_{\mathcal{I}}(v_t^{(j)})\|^2] &\leq 4L_s(f(x^*) - f(x_t^{(j)})) + 4L_s(f(x_0^{(j)}) - f(x^*)) + 4L_s(\langle \pi_{\mathcal{I}}(\nabla f(x_t^{(j)})), x_t^{(j)} - x^*\rangle) \\ &\quad + 2\|\pi_{\mathcal{I}}(\nabla f(x^*))\|^2 + 2\|\pi_{\mathcal{I}}(\nabla f(x_t^{(j)}))\|^2 + 2L_s^2\|x_0^{(j)} - x^*\|^2 + 2\|\pi_{\mathcal{I}}(e^{(j)})\|^2 + E[\|\pi_{\mathcal{I}}(u_t^{(j)})\|^2] \end{aligned} \tag{7}$$

where $e^{(j)}$ is the bias of $v_t^{(j)}$. The Eq. (7) indicates that the variance of stochastic gradient direction can diminish to zero, when the model estimator $x^{(j)}$ is approaching to optimal x^* and x^* is close to solution of the unconstrained problem (1), as long as both $\|\pi_{\mathcal{I}}(e^{(j)})\|^2$ and $E[\|\pi_{\mathcal{I}}(u_t^{(j)})\|^2]$ are small.

2) Proof. With above preparations, we are ready to give the logic line of proof for main theorem. In order to analyze the DP-SCSG-HT algorithm, we develop the following result,

$$E_{I_t^{(j)}}[\|\tilde{x}_{t+1}^{(j)} - x^*\|^2] = E_{I_t^{(j)}}[\|x_t^{(j)} - x^*\|^2] + \eta^2 E_{I_t^{(j)}}[\|\pi_{\mathcal{I}}(v_t^{(j)})\|^2] - 2\eta\langle \pi_{\mathcal{I}}(\nabla f(x_t^{(j)})), x_t^{(j)} - x^*\rangle - 2\eta\langle \pi_{\mathcal{I}}(e^{(j)}), x_t^{(j)} - x^*\rangle$$

where $\tilde{x}_{t+1}^{(j)} = x_t^{(j)} - \eta\pi_{\mathcal{I}}(v_t^{(j)})$ is an intermediate state of the estimator to bridge the analysis between the gradient-based updating step and the hard thresholding step. Then the hard thresholding operation $x_{t+1}^{(j)} = \mathcal{H}_k(\tilde{x}_{t+1}^{(j)})$ immediately follows and we can get $x_{t+1}^{(j)} = \mathcal{H}_k(x_t^{(j)} - \eta v_t^{(j)})$ due to $\mathcal{I} = supp(x^*) \cup supp(x_t^{(j)}) \cup supp(x_{t+1}^{(j)})$.

Next, we establish connections between the intermediate state $\tilde{x}_{t+1}^{(j)}$ and the sparse estimator $x_{t+1}^{(j)}$, by Lemma 2.6, we get

$$\begin{aligned} E^{(j)}[\|x_{t+1}^{(j)} - x^*\|^2] &\leq (1+\beta)E^{(j)}[\|\tilde{x}_{t+1}^{(j)} - x^*\|^2] \\ &\leq (1+\beta)E^{(j)}[\|x_t^{(j)} - x^*\|^2] + (1+\beta)\eta^2 E^{(j)}[\|\pi_{\mathcal{I}}(v_t^{(j)})\|^2] - 2(1+\beta)\eta E^{(j)}[\langle \pi_{\mathcal{I}}(\nabla f(x_t^{(j)})), x_t^{(j)} - x^*\rangle]. \end{aligned} \tag{8}$$

Until now, all the analyses are still based on iterations in one epoch. We need to use an important property of the geometric distribution that we have used to set the number of inner iterations $N^{(j)}$ to turn previous iteration-based analysis into the epoch-based analysis. Let $N \sim Geom(\gamma)$, for any sequence $\{D_N\}$, we have $E[D_N - D_{N+1}] = (\frac{1}{\gamma} - 1)(D_0 - E[D_N])$. Taking the expectation on both sides of Eq. (8) over $N^{(j)}$, and replacing $x_0^{(j)}$ with $\tilde{x}^{(j)}$ and $x_{N^{(j)}}^{(j)}$ with $\tilde{x}^{(j+1)}$ yields the most important intermediate result:

$$\begin{aligned} &2(1+\beta)\eta E[\langle \pi_{\mathcal{I}}(\nabla f(\tilde{x}^{(j+1)})), \tilde{x}^{(j+1)} - x^*\rangle] \\ &\leq (\beta - \frac{b}{B})E[\|\tilde{x}^{(j+1)} - x^*\|^2] + \frac{b}{B}E[\|\tilde{x}^{(j)} - x^*\|^2] + (1+\beta)\eta^2 E[\|\pi_{\mathcal{I}}(v_{N^{(j)}}^{(j)})\|^2]. \end{aligned} \tag{9}$$

After obtaining the above results, we put Eq. (7) for $E[\|\pi_{\mathcal{I}}(v_{N^{(j)}}^{(j)})\|^2]$ into Eq. (9) and further using ρ_s-restricted strongly convex and L_s-restricted strongly smooth, we obtain the desired result. □

Remark 4.4. *Due to the requirement on β that $\frac{B}{b} \leq \frac{1}{\beta} = \frac{\sqrt{k-k^*}}{2\sqrt{k^*}} = \Theta(\sqrt{k})$, $\frac{B}{b}$ is independent of the sample size n. Hence, $\frac{B}{b}$ can be treated as a constant independent of ϵ and δ, and be omitted in the asymptotic utility bound in the next sections.*

The implication of the main theorem is that the variance of stochastic gradients σ_0^2 can be well-controlled by the batch size B, and the requirement for the upper bound of the stochastic variance σ_0^2 will be relaxed with the increase of B and be removed when $B = n$. Therefore, unlike the DP-SGD-HT, there is no need to bound σ_0^2 in Algorithm 2 with $B = n$. Nevertheless, a careful setup for B could save the number of IFO calls. Even though setting $B = n$ could fully remove σ_0^2, it needs to be carefully designed to achieve the best of the two worlds, which means that the effect of σ_0^2 is minimized to the distance bound between estimator x_T and optimal x^*, and the batch size B is also minimized to achieve such goal to avoid the waste of computations.

Similar to the analysis of the DP-SGD-HT, the two bias terms in the parameter estimation of DP-SCSG-HT are the second term and third term of Eq. (5): $O(\frac{\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\|^2}{\kappa_s^2} + \frac{\sigma^2}{\kappa_s^2})$. Compared to the bias of the DP-SGD-HT $O(\left\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\right\|^2 + \sigma^2)$, the bias of the DP-SCSG-HT shrinks by a factor of κ_s^2. The value of κ_s^2 is > 1 and can be very large for ill-conditioned optimization problems. As discussed in Remark 4.2, the variance of Gaussian noise is also smaller in the DP-SCSG-HT than in the DP-SGD-HT in practice. Hence, the DP-SCSG-HT tends to have smaller bias in terms of parameter estimation.

4.3. The Utility Bound of the DP-SCSG-HT

With the upper bound between $\tilde{x}^{(\mathcal{J})}$ and the optimal x^* in Theorem 4.3, and the determined Gaussian variance σ^2 in Theorem 4.1, we obtain the utility bound as follows.

Theorem 4.5 (Utility)**.** *Under the same setting of Theorem 4.3, and* $B = \max\{1, \sqrt{\frac{2b\epsilon\sigma_0^2}{3k\alpha C\mathcal{J}l^2}}\}\cdot n$, *if we choose* $\mathcal{J} = O(\log(\frac{n^2\epsilon^2}{\log(1/\delta)}))$, *we get*

$$E[\|\tilde{x}^{(\mathcal{J})} - x^*\|^2] \le \frac{8\eta^2}{(1-\theta_2)\gamma}\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\|^2 + O(\frac{\log(1/\delta)}{n^2\epsilon^2}\log(\frac{n^2\epsilon^2}{\log(1/\delta)})). \tag{10}$$

Proof. If we require that $\mathbb{I}(B < n)\sigma_0^2 \le kB\sigma^2$ and $\sigma^2 = \frac{2CBl^2\alpha\mathcal{J}}{bn^2\epsilon}$, we have $B = \min\{1, \sqrt{\frac{2b\epsilon\sigma_0^2}{3k\alpha C\mathcal{J}l^2}}\} * n$.

$$\begin{aligned} E[\|\tilde{x}^{(\mathcal{J})} - x^*\|^2] &\le \theta_2^{\mathcal{J}} E[\|\tilde{x}^{(0)} - x^*\|^2] + \frac{8\eta^2}{(1-\theta_2)\gamma}E[\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\|^2] + \frac{7k\eta^2\sigma^2}{(1-\theta_2)\gamma} \\ &= \theta_2^{\mathcal{J}}\|\tilde{x}^{(0)} - x^*\|^2 + \frac{8\eta^2}{(1-\theta_2)\gamma}E[\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\|^2] + \frac{1}{1-\theta_2}\frac{7k\eta^2}{\gamma}\frac{2CBl^2\alpha\mathcal{J}}{bn^2\epsilon}. \end{aligned}$$

If we let $\theta_2^{\mathcal{J}}\|\tilde{x}^{(0)} - x^*\|^2 \le \frac{1}{1-\theta_2}\frac{7k\eta^2}{\gamma}\frac{2CBl^2\alpha}{bn^2\epsilon}$ and $\frac{B}{b} = \Theta(\sqrt{k})$, we get

$$\mathcal{J} = \log_{\theta_2}(\frac{1}{(1-\theta_2)\|\tilde{x}^{(0)} - x^*\|^2}\frac{7k\eta^2}{\gamma}\frac{2CBl^2\alpha}{bn^2\epsilon}) = O(\log(\frac{n^2\epsilon^2}{\log(1/\delta)})).$$

Finally, we get

$$E[\|\tilde{x}^{(\mathcal{J})} - x^*\|^2] \le \frac{8\eta^2}{(1-\theta_2)\gamma}\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\|^2 + O(\frac{\log(1/\delta)}{n^2\epsilon^2}\log(\frac{n^2\epsilon^2}{\log(1/\delta)})).$$

□

Considering that $\|\pi_{\tilde{\mathcal{I}}}(\nabla f(x^*))\|^2$ can be close to zero, when x^* is close to its unconstrained optional for $f(x)$, Theorem 4.5 implies that the utility bound is determined by its dominant term in the order of $O(\frac{\log(1/\delta)}{n^2\epsilon^2})$, which achieves the same utility guarantee with DP-GD-HT. Furthermore, the next corollary shows better computationally complexity of our proposed practical stochastic variance reduced algorithm.

Corollary 4.5.1 (Computational Complexity)**.** *Under the same conditions of Theorem 4.5, the number of IFO calls is* $O(\min\{1, \psi\}\cdot n\log(\frac{n^2\epsilon^2}{\log(1/\delta)}))$, *where* $\psi = \sqrt{\frac{2b\epsilon\sigma_0^2}{3k\alpha C\mathcal{J}l^2}}$.

To obtain a given $(\epsilon, \delta)-$DP, the computational complexity of DP-SCSG-HT depends on $O(\min\{1, \psi\}\cdot n\log(n))$. Because $\sigma_0^2 \le l^2$, $\alpha > 1$ and $C > 1$, ψ can be much smaller than 1, if sparsity k and epoch size $\mathcal{J}$ are large, batch size b is small (it is especially true for high dimensional data). Therefore, similar to DP-SGD-HT, DP-SCSG-HT can be much more computationally efficient than DP-GD-HT, which means fewer number of epochs are used in Algorithm 2 to achieve the same $(\epsilon, \delta)-$DP.

In summary, our proposed algorithm provides a general framework, which covers the existing state-of-the-art non-DP hard thresholding method: SVRG-HT [32] (when $B = n$, $b = 1$, $\sigma^2 = 0$ and option II is selected), which corresponding to privacy-preserving version can be called as DP-SVRG-HT. Even though we only use option I to theoretically analyze Algorithm 2 for clarity, our DP guarantee can be directly applied to option II and so is to DP-SVRG-HT. Following the line of proof above, the convergence analysis for DP-SVRG-HT can be done, and then utility bound can be built in the same way in section 4.3.

5. Empirical Evaluations

In this section, we compare the proposed stochastic privacy-preserving algorithms DP-SGD-HT and DP-SCSG-HT with the state-of-the-art deterministic sparsity-constraint method: DP-GD-HT[56], to demonstrate the improved performance and advantage of the stochastic methods. DP-GD-HT has been implemented based on the design in [56] and applied to our experimental datasets. Moreover, our DP-SCSG-HT, if removing the Gaussian noise perturbation to the gradients, is a non-DP method including SVRG-HT[32] and SCSG-HT[33] as special cases. We hence use it to report non-DP baseline performance. Note that the non-DP methods are expected to produce better accuracy performance given they are not constrained to satisfy DP.

5.1. Experimental Setup

Two benchmark datasets, E2006-tfidf and RCV1, are downloaded from the LibSVM website[1] , and used for evaluation. The E2006-tfidf dataset [26] has 3,308 observations, each described by 150,360 features, to predict the volatility of stock returns based on the mandated financial text report. Data have been collected from thousands of publicly traded U.S. companies. The RCV1 dataset [30] contains 20,242 observations and 47,236 features, and is used to predict categories of newswire stories recently collected by Reuters. Ltd. We also conduct experiments on the medical data: Chest X-ray [24], which has 5232 observations and 784 features, and is for Pneumonia Detection. In the experiments, the variance of the injected random noise in the different algorithms is chosen according to the suggested values in their theoretical results. Other parameters, such as the batch size, stepsize and number of epochs, are determined by five-fold cross-validation. Particularly, the stepsize η for each algorithm is searched from $\{10, 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$ and number of epoch is searched from $\{10, 20, 50, 80, 100\}$. All the algorithms are initialized with $x^{(0)} = 0$. Following the convention in the stochastic optimization and sparse learning literature, we use the number of epochs (or data passes) to measure the computational complexity. This enables the complexity study independent of an actual implementation of the algorithm. All experiments are done on PC with i7-6700 CPU, 4 cores, 8GB RAM.

5.2. Linear Regression

We first conduct experiments on the linear regression problem

$$\min_x \{f(x) = \frac{1}{n}\sum_{i=1}^{n} \|y_i - z_i^T x\|^2\} \text{ subject to } \|x\|_0 \le k,$$

to check the performance of the proposed DP-SGD-HT and DP-SCSG-HT algorithms. The dataset we use is the E2006-tfidf dataset [26]. In the experiments, we set the sparsity parameter $k = 200$, $\delta = 10^{-5}$ and $\epsilon \in [2, 10]$. Table 2 compares the mean squared errors (MSE) of the different methods on validation data under different choices of privacy budget ϵ. In a five-fold cross-validation process, the MSE values are averaged across the five validation sets together with standard deviation. Precisely, MSE on a single validation set is defined as follows: $\frac{1}{n_{val}}\|Z_{val}^T\tilde{x} - y_{val}\|^2$, where $\{Z_{val}, y_{val}\}$ are the validation data, n_{val} is the validation sample size and $\tilde{x}$ is the estimator learned from the training data. The results in Table 2 show that under the same guarantee of (ϵ, δ)-DP, the proposed methods: DP-SGD-HT and DP-SCSG-HT achieve lower MSE using a smaller number of epochs than the DP-GD-HT. Therefore, the utility

[1] http://www.csie.ntu.edu.tw/ cjlin/libsvmtools/datasets/

Table 2: Comparisons of different algorithms for various privacy budgets ϵ in terms of MSE on the validation data of five-fold cross validation and its corresponding standard deviation on the dataset E2006-tfidf. Note that $\delta = 10^{-5}$ in the experiment. The non-DP Baseline is obtained by the a special case of SCSG-HT: SCSG-HT [33], which is the state-of-the-art of non-DP IHT algorithms. Each column represents one group of experiment for fixed privacy guarantee (ϵ, δ)-DP. Epoch is the measure for computational complexity. The results show that DP-SCSG-HT achieves the lowest MSE among DP algorithms and is closer to the non-private baseline.

Methods	Epoch	Differential private budget ϵ				
		$\epsilon = 2$	$\epsilon = 4$	$\epsilon = 6$	$\epsilon = 8$	$\epsilon = 10$
Non-DP Baseline [33]	10	0.1483 ± 0.013	0.1483 ± 0.013	0.1483 ± 0.013	0.1483 ± 0.013	0.1483 ± 0.013
DP-GD-HT[56]	100	0.1588 ± 0.025	0.1566 ± 0.015	0.1560 ± 0.009	0.1543 ± 0.01	0.1528 ± 0.012
DP-SGD-HT	20	0.1540 ± 0.005	0.1505 ± 0.009	0.1499 ± 0.013	0.1488 ± 0.011	0.1490 ± 0.013
DP-SCSG-HT	10	$\mathbf{0.1516 \pm 0.007}$	$\mathbf{0.1494 \pm 0.014}$	$\mathbf{0.1488 \pm 0.007}$	$\mathbf{0.1487 \pm 0.006}$	$\mathbf{0.1486 \pm 0.012}$

Table 3: Comparisons of different algorithms for various privacy budgets ϵ in terms of the validation loss (11) on validation data of five-fold cross validation and its corresponding standard deviation on dataset RCV1. Note that $\delta = 10^{-5}$ in the experiment. The results show that DP-SCSG-HT achieves the lowest validation loss with the smallest number of epochs, among DP methods.

Methods	Epoch	Differential private budget ϵ				
		$\epsilon = 2$	$\epsilon = 4$	$\epsilon = 6$	$\epsilon = 8$	$\epsilon = 10$
Non-DP Baseline [33]	10	0.1603 ± 0.003	0.1603 ± 0.003	0.1603 ± 0.003	0.1603 ± 0.003	0.1603 ± 0.003
DP-GD-HT[56]	100	0.3811 ± 0.011	0.3469 ± 0.01	0.3139 ± 0.004	0.3063 ± 0.009	0.3001 ± 0.004
DP-SGD-HT	20	0.4100 ± 0.027	0.2914 ± 0.012	0.2594 ± 0.008	0.2615 ± 0.011	0.2491 ± 0.008
DP-SCSG-HT	10	$\mathbf{0.2243 \pm 0.012}$	$\mathbf{0.1662 \pm 0.022}$	$\mathbf{0.1642 \pm 0.007}$	$\mathbf{0.1619 \pm 0.002}$	$\mathbf{0.1615 \pm 0.005}$

Table 4: Comparisons of different algorithms for various privacy budgets ϵ in terms of the validation loss (11) on validation data of five-fold cross validation and its corresponding standard deviation on dataset Chest X-ray. Note that $\delta = 10^{-5}$ in the experiment. The results show that DP-SCSG-HT achieves the lowest validation loss with the smallest number of epochs, among DP methods, except $\epsilon = 2$.

Methods	Epoch	Differential private budget ϵ				
		$\epsilon = 2$	$\epsilon = 4$	$\epsilon = 6$	$\epsilon = 8$	$\epsilon = 10$
Non-DP Baseline [33]	10	0.6917 ± 0.007	0.6917 ± 0.007	0.6917 ± 0.007	0.6917 ± 0.007	0.6917 ± 0.007
DP-GD-HT[56]	100	$\mathbf{0.6930 \pm 0.003}$	0.6929 ± 0.001	0.6928 ± 0.004	0.6928 ± 0.002	0.6927 ± 0.002
DP-SGD-HT	20	0.6932 ± 0.007	$\mathbf{0.6927 \pm 0.002}$	0.6926 ± 0.003	0.6924 ± 0.001	0.6922 ± 0.004
DP-SCSG-HT	10	0.6940 ± 0.002	0.6928 ± 0.004	$\mathbf{0.6919 \pm 0.008}$	$\mathbf{0.6918 \pm 0.001}$	$\mathbf{0.6918 \pm 0.001}$

and computational complexity of our stochastic methods are better than that of the non-stochastic DP-GD-HT. DP algorithms take the balance between privacy-preserving degree and optimization accuracy, but our algorithms exhibit better accuracy even under the DP requirement.

5.3. Logistic Regression

Then, we apply all methods to the logistic regression problem as follows

$$\min_x \{f(x) = \frac{1}{n}\sum_{i=1}^{n}(\log(1 + exp(y_i z_i^T x)) + \frac{\lambda}{2}\|x\|^2)\} \text{ subject to } \|x\|_0 \le k,$$

where $z_i \in \mathbb{R}^d$ and y_i is the corresponding label. The dataset we use are the RCV1 dataset [30] and Chest X-ray[24]. For experiment with RCV1, the regularizer $\lambda = 10^{-5}$ and the sparsity parameter $k = 1000$. For experiment with Chest X-ray, the regularizer $\lambda = 10^{-5}$ and the sparsity parameter $k = 200$. We use five-fold cross-validation to calculate the value of loss function:

$$\frac{1}{n_{val}} \sum_{i=1}^{n_{val}} (\log(1 + exp(y_i z_i^T x)) \tag{11}$$

over validation data and our proposed algorithm DP-SCSG-HT could achieve the lowest loss value among all privacy-preserving algorithms on the RCV1 data in Table 4.

Separate from the five-fold cross validation, we run all algorithms on the full dataset so to compare the computational efficiency of the different algorithms. We demonstrate the advantage of the stochastic algorithms by plotting the objective function value $f(x)$ versus the number of epochs and the number of hard thresholding operations of different algorithms at the privacy budget $\epsilon \in \{4, 6\}$ on RCV1 in Figure 1 and Chest X-ray in Figure 2 . Our algorithms outperform the deterministic DP-GD-HT in terms of the needed epochs by a large margin, which is consistent with our theoretical results. While DP-SCSG-HT and DP-SGD-HT achieve the best results within 20 epochs, DP-GD-HT needs much more epochs. Therefore, the proposed algorithms could drop objective function value more rapidly, while guaranteeing the $(\epsilon, \delta)-$DP.

Remark 5.1. *The guarantee of (ϵ, δ)-DP adds an extra constraint to the optimization algorithm; Therefore, DP algorithms preserve privacy at the cost of losing prediction accuracy (utility). Empirically, we did observe that our algorithms slightly sacrifice prediction accuracy, and the performance gap with the non-DP baseline is reduced with larger privacy budget ϵ by using a smaller level of injected Gaussian noise. Hence, the algorithms play balance between privacy preserving and utility. The observations are consistent with prior discussions in [56] .*

6. Conclusions

In this paper, this work is mainly concerned with the utility and computational complexity of a set of sparse learning algorithms that satisfy DP. We propose two privacy-preserving iterative stochastic gradient hard thresholding algorithms for sparse learning: DP-SGD-HT and DP-SCSG-HT. To balance between DP and the algorithmic utility, the proposed algorithms play trade-off between the magnitude of perturbation noise and the privacy budget. The higher the perturbation noise, the more privacy preserving but less algorithm utility. We establish a linear convergence rate for both algorithms under certain bias introduced by sparsity and DP requirement. Our algorithms also achieve the best known utility bound, and meanwhile they significantly reduce computational complexity from the GD based algorithm. We emphasize that although the convergence proof of DP-SGD-HT requires the variance of stochastic gradients to be bound which is however removed in the convergence proof of DP-SCSG-HT. Experiments on real-world financial and medical datasets demonstrate the superiority of our proposed algorithm against state-of-the-art baseline algorithms.

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References

[1] M. Abadi, A. Chu, I. Goodfellow, H. B. McMahan, I. Mironov, K. Talwar, and L. Zhang. Deep learning with differential privacy. In *Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security*, pages 308–318. ACM, 2016.

[2] A. Agarwal and L. Bottou. A lower bound for the optimization of finite sums. *arXiv preprint arXiv:1410.0723*, 2014.

[3] Z. Allen-Zhu. Katyusha x: Practical momentum method for stochastic sum-of-nonconvex optimization. *arXiv preprint arXiv:1802.03866*, 2018.

[4] D. Apple. Learning with privacy at scale. *Apple Machine Learning Journal*, 1(8), 2017.

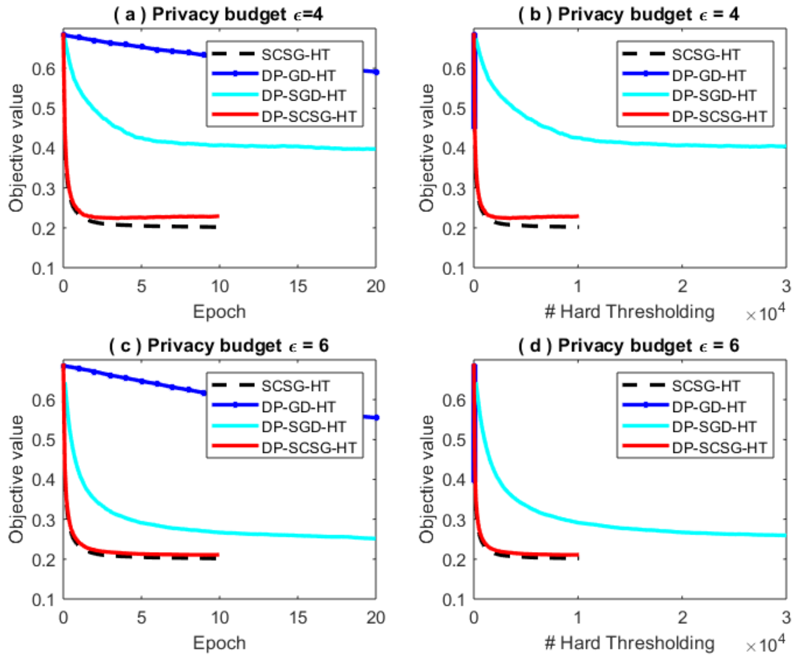

Figure 1: Experimental results for logistic regression with sparsity constraint on the RCV1 data. Figures (a-b) show results for the $(4, 10^{-5})$-DP and (c-d) for the $(6, 10^{-5})$-DP. (a) and (c) show the objective value $f(x)$ on the full dataset versus the number of epoch. (b), (d) the objective value $f(x)$ on the full dataset versus the number of HT operations.

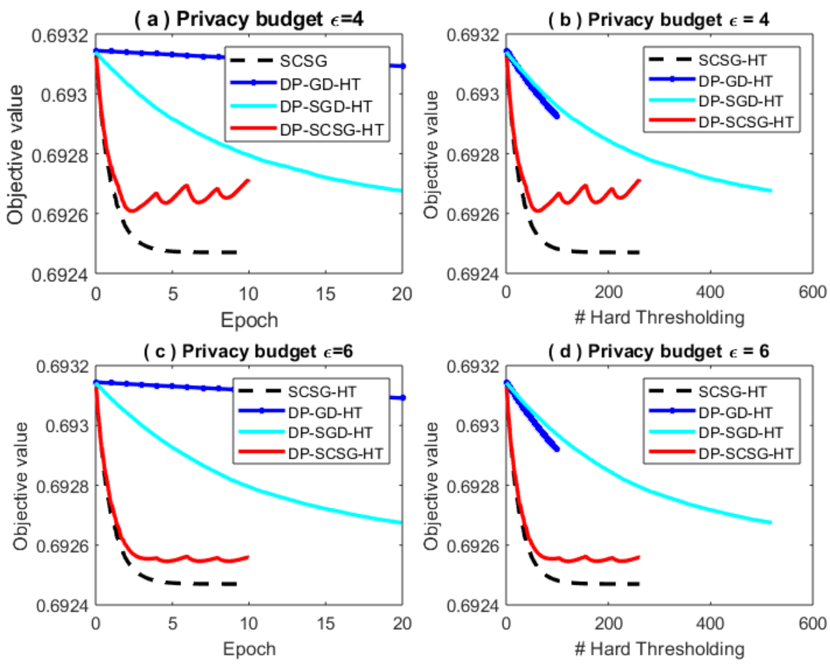

Figure 2: Experimental results for logistic regression with sparsity constraint on the Chest X-ray data. Figures (a-b) show results for the $(4, 10^{-5})$-DP and (c-d) for the $(6, 10^{-5})$-DP. (a) and (c) show the objective value $f(x)$ on the full dataset versus the number of epoch. (b), (d) the objective value $f(x)$ on the full dataset versus the number of HT operations.

[5] S. Bahmani, B. Raj, and P. T. Boufounos. Greedy sparsity-constrained optimization. *Journal of Machine Learning Research*, 14(Mar): 807–841, 2013.

[6] R. Bassily, A. Smith, and A. Thakurta. Private empirical risk minimization: Efficient algorithms and tight error bounds. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, pages 464–473. IEEE, 2014.

[7] T. Blumensath and M. E. Davies. Iterative hard thresholding for compressed sensing. *Applied and computational harmonic analysis*, 27(3): 265–274, 2009.

[8] K. Chaudhuri, C. Monteleoni, and A. D. Sarwate. Differentially private empirical risk minimization. *Journal of Machine Learning Research*, 12(Mar):1069–1109, 2011.

[9] A. E. C. Cloud. Amazon web services. *Retrieved November*, 9(2011):2011, 2011.

[10] A. Defazio, F. Bach, and S. Lacoste-Julien. Saga: A fast incremental gradient method with support for non-strongly convex composite objectives. In *Advances in neural information processing systems*, pages 1646–1654, 2014.

[11] D. L. Donoho et al. Compressed sensing. *IEEE Transactions on information theory*, 52(4):1289–1306, 2006.

[12] C. Dwork, K. Kenthapadi, F. McSherry, I. Mironov, and M. Naor. Our data, ourselves: Privacy via distributed noise generation. In *Annual International Conference on the Theory and Applications of Cryptographic Techniques*, pages 486–503. Springer, 2006.

[13] C. Dwork, F. McSherry, K. Nissim, and A. Smith. Calibrating noise to sensitivity in private data analysis. In *Theory of cryptography conference*, pages 265–284. Springer, 2006.

[14] C. Dwork, G. N. Rothblum, and S. Vadhan. Boosting and differential privacy. In *2010 IEEE 51st Annual Symposium on Foundations of Computer Science*, pages 51–60. IEEE, 2010.

[15] M. Elibol, L. Lei, and M. I. Jordan. Variance reduction with sparse gradients. *arXiv preprint arXiv:2001.09623*, 2020.

[16] Ú. Erlingsson, V. Pihur, and A. Korolova. Rappor: Randomized aggregatable privacy-preserving ordinal response. In *Proceedings of the 2014 ACM SIGSAC conference on computer and communications security*, pages 1054–1067, 2014.

[17] C. Fang, C. J. Li, Z. Lin, and T. Zhang. Spider: Near-optimal non-convex optimization via stochastic path-integrated differential estimator. In *Advances in Neural Information Processing Systems*, pages 686–696, 2018.

[18] S. Foucart. Hard thresholding pursuit: an algorithm for compressive sensing. *SIAM Journal on Numerical Analysis*, 49(6):2543–2563, 2011.

[19] R. B. Harikandeh, M. O. Ahmed, A. Virani, M. Schmidt, J. Konečnỳ, and S. Sallinen. Stopwasting my gradients: Practical svrg. In *Advances in Neural Information Processing Systems*, pages 2251–2259, 2015.

[20] P. Jain, A. Tewari, and P. Kar. On iterative hard thresholding methods for high-dimensional m-estimation. In *Advances in Neural Information Processing Systems*, pages 685–693, 2014.

[21] A. Jalali, C. C. Johnson, and P. K. Ravikumar. On learning discrete graphical models using greedy methods. In *Advances in Neural Information Processing Systems*, pages 1935–1943, 2011.

[22] R. Johnson and T. Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In *Advances in neural information processing systems*, pages 315–323, 2013.

[23] S. P. Kasiviswanathan, H. K. Lee, K. Nissim, S. Raskhodnikova, and A. Smith. What can we learn privately? *SIAM Journal on Computing*, 40(3):793–826, 2011.

[24] D. S. Kermany, M. Goldbaum, W. Cai, C. C. Valentim, H. Liang, S. L. Baxter, A. McKeown, G. Yang, X. Wu, F. Yan, et al. Identifying medical diagnoses and treatable diseases by image-based deep learning. *Cell*, 172(5):1122–1131, 2018.

[25] D. Kifer, A. Smith, and A. Thakurta. Private convex empirical risk minimization and high-dimensional regression. In *Conference on Learning Theory*, pages 25–1, 2012.

[26] S. Kogan, D. Levin, B. R. Routledge, J. S. Sagi, and N. A. Smith. Predicting risk from financial reports with regression. In *Proceedings of Human Language Technologies: The 2009 Annual Conference of the North American Chapter of the Association for Computational Linguistics*, pages 272–280, 2009.

[27] L. Lei and M. Jordan. Less than a single pass: Stochastically controlled stochastic gradient. In *Artificial Intelligence and Statistics*, pages 148–156, 2017.

[28] L. Lei and M. Jordan. Less than a single pass: Stochastically controlled stochastic gradient. In *Artificial Intelligence and Statistics*, pages 148–156, 2017.

[29] L. Lei, C. Ju, J. Chen, and M. I. Jordan. Non-convex finite-sum optimization via scsg methods. In *Advances in Neural Information Processing Systems*, pages 2348–2358, 2017.

[30] D. D. Lewis, Y. Yang, T. G. Rose, and F. Li. Rcv1: A new benchmark collection for text categorization research. *Journal of machine learning research*, 5(Apr):361–397, 2004.

[31] X. Li, R. Arora, H. Liu, J. Haupt, and T. Zhao. Nonconvex sparse learning via stochastic optimization with progressive variance reduction. *arXiv preprint arXiv:1605.02711*, 2016.

[32] X. Li, T. Zhao, R. Arora, H. Liu, and J. Haupt. Stochastic variance reduced optimization for nonconvex sparse learning. In *International Conference on Machine Learning*, pages 917–925, 2016.

[33] G. Liang, Q. Tong, C. J. Zhu, and J. Bi. An effective hard thresholding method based on stochastic variance reduction for nonconvex sparse learning. In *AAAI*, pages 1585–1592, 2020.

[34] S. G. Mallat and Z. Zhang. Matching pursuits with time-frequency dictionaries. *IEEE Transactions on signal processing*, 41(12):3397–3415, 1993.

[35] L. Melis, C. Song, E. De Cristofaro, and V. Shmatikov. Exploiting unintended feature leakage in collaborative learning. In *2019 IEEE S&P*, pages 691–706, San Francisco, CA, 2019. IEEE.

[36] I. Mironov. Rényi differential privacy. In *2017 IEEE 30th Computer Security Foundations Symposium (CSF)*, pages 263–275. IEEE, 2017.

[37] S. Mohamed, K. Heller, and Z. Ghahramani. Bayesian and l1 approaches to sparse unsupervised learning. *arXiv preprint arXiv:1106.1157*, 2011.

[38] B. K. Natarajan. Sparse approximate solutions to linear systems. *SIAM journal on computing*, 24(2):227–234, 1995.

[39] J. Near. Differential privacy at scale: Uber and berkeley collaboration. In *Enigma 2018 (Enigma 2018)*, 2018.

[40] D. Needell and J. A. Tropp. Cosamp: Iterative signal recovery from incomplete and inaccurate samples. *Applied and computational harmonic*

analysis, 26(3):301–321, 2009.

[41] L. M. Nguyen, J. Liu, K. Scheinberg, and M. Takáč. Sarah: A novel method for machine learning problems using stochastic recursive gradient. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 2613–2621. JMLR. org, 2017.

[42] N. Nguyen, D. Needell, and T. Woolf. Linear convergence of stochastic iterative greedy algorithms with sparse constraints. *IEEE Transactions on Information Theory*, 63(11):6869–6895, 2017.

[43] N. Papernot, M. Abadi, U. Erlingsson, I. Goodfellow, and K. Talwar. Semi-supervised knowledge transfer for deep learning from private training data. *arXiv preprint arXiv:1610.05755*, 2016.

[44] Y. C. Pati, R. Rezaiifar, and P. S. Krishnaprasad. Orthogonal matching pursuit: Recursive function approximation with applications to wavelet decomposition. In *Proceedings of 27th Asilomar conference on signals, systems and computers*, pages 40–44. IEEE, 1993.

[45] A. Quattoni, M. Collins, and T. Darrell. Transfer learning for image classification with sparse prototype representations. In *2008 IEEE Conference on Computer Vision and Pattern Recognition*, pages 1–8. IEEE, 2008.

[46] S. Ravishankar and Y. Bresler. Learning sparsifying transforms. *IEEE Transactions on Signal Processing*, 61(5):1072–1086, 2012.

[47] S. J. Reddi, A. Hefny, S. Sra, B. Poczos, and A. Smola. Stochastic variance reduction for nonconvex optimization. In *International conference on machine learning*, pages 314–323, 2016.

[48] A. Rényi et al. On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*. The Regents of the University of California, 1961.

[49] R. Shokri, M. Stronati, C. Song, and V. Shmatikov. Membership inference attacks against machine learning models. In *2017 IEEE S&P*, pages 3–18, San Jose, CA, 2017. IEEE.

[50] K. Talwar, A. G. Thakurta, and L. Zhang. Nearly optimal private lasso. In *Advances in Neural Information Processing Systems*, pages 3025–3033, 2015.

[51] A. G. Thakurta and A. Smith. Differentially private feature selection via stability arguments, and the robustness of the lasso. In *Conference on Learning Theory*, pages 819–850, 2013.

[52] J. A. Tropp and A. C. Gilbert. Signal recovery from random measurements via orthogonal matching pursuit. *IEEE Transactions on information theory*, 53(12):4655–4666, 2007.

[53] S. Truex, L. Liu, M. E. Gursoy, L. Yu, and W. Wei. Demystifying membership inference attacks in machine learning as a service. *IEEE Transactions on Services Computing*, 2019.

[54] D. Wang and J. Xu. On sparse linear regression in the local differential privacy model. In *International Conference on Machine Learning*, pages 6628–6637, 2019.

[55] D. Wang, M. Ye, and J. Xu. Differentially private empirical risk minimization revisited: Faster and more general. In *Advances in Neural Information Processing Systems*, pages 2722–2731, 2017.

[56] L. Wang and Q. Gu. Differentially private iterative gradient hard thresholding for sparse learning. In *28th International Joint Conference on Artificial Intelligence*, 2019.

[57] L. Wang and Q. Gu. A knowledge transfer framework for differentially private sparse learning. *arXiv preprint arXiv:1909.06322*, 2019.

[58] L. Wang, B. Jayaraman, D. Evans, and Q. Gu. Efficient privacy-preserving nonconvex optimization. *arXiv preprint arXiv:1910.13659*, 2019.

[59] Y.-X. Wang, B. Balle, and S. Kasiviswanathan. Subsampled r\'enyi differential privacy and analytical moments accountant. *arXiv preprint arXiv:1808.00087*, 2018.

[60] S. J. Wright and J. Nocedal. Numerical optimization. *Springer Science*, 35(67-68):7, 1999.

[61] X. Wu, F. Li, A. Kumar, K. Chaudhuri, S. Jha, and J. Naughton. Bolt-on differential privacy for scalable stochastic gradient descent-based analytics. In *Proceedings of the 2017 ACM International Conference on Management of Data*, pages 1307–1322, 2017.

[62] L. Xie, K. Lin, S. Wang, F. Wang, and J. Zhou. Differentially private generative adversarial network. *arXiv preprint arXiv:1802.06739*, 2018.

[63] J. Zhang, K. Zheng, W. Mou, and L. Wang. Efficient private erm for smooth objectives. *arXiv preprint arXiv:1703.09947*, 2017.

[64] P. Zhou, X. Yuan, and J. Feng. Efficient stochastic gradient hard thresholding. In *Advances in Neural Information Processing Systems*, pages 1988–1997, 2018.

[65] J. Zhu, X. Fang, Z. Guo, M. H. Niu, F. Cao, S. Yue, and Q. Y. Liu. Ibm cloud computing powering a smarter planet. In *IEEE International Conference on Cloud Computing*, pages 621–625. Springer, 2009.